Derivative corrections to the Born-Infeld action through beta-function calculations in N=2 "boundary" superspace

Stijn Nevens and Alexander Sevrin

Theoretische Natuurkunde, Vrije Universiteit Brussel and
The International Solvay Institutes
Pleinlaan 2, B-1050 Brussels, Belgium
E-mail: stijn@tena4.vub.ac.be, Alexandre.Sevrin@vub.ac.be

Walter Troost

Instituut voor Theoretische Fysica, Katholieke Universiteit Leuven
Celestijnenlaan 200D, B-3001 Leuven, Belgium.
E-mail: Walter.Troost@fys.kuleuven.ac.be

AlexanderWijns[*]

Theoretische Natuurkunde, Vrije Universiteit Brussel and
The International Solvay Institutes
Pleinlaan 2, B-1050 Brussels, Belgium
E-mail: awijns@tena4.vub.ac.be

ABSTRACT: We calculate the β-functions for an open string σ-model in the presence of a U(1) background. Passing to $N = 2$ boundary superspace, in which the background is fully characterized by a scalar potential, significantly facilitates the calculation. Performing the calculation through three loops yields the equations of motion up to five derivatives on the fieldstrengths, which upon integration gives the bosonic sector of the effective action for a single D-brane in trivial bulk background fields through four derivatives and to all orders in α'. Finally, the present calculation shows that demanding ultra-violet finiteness of the non-linear σ-model can be reformulated as the requirement that the background is a deformed stable holomorphic U(1) bundle.

[*]Aspirant FWO

Contents

1. Introduction

The effective world volume action for n coinciding Dp-branes is, in leading order in α', given by the $d = 9 + 1$, $N = 1$ supersymmetric U(n) Yang-Mills action dimensionally reduced to $p + 1$ dimensions [1]. For a single D-brane, $n = 1$, the effective action is known to all orders in α' in the limit of constant (or slowly varying) background fields: it is the $d = 9 + 1$, $N = 1$ supersymmetric Born-Infeld action, dimensionally reduced to $p + 1$ dimensions, [2] – [10]. Both bosonic and fermionic terms as well as the couplings to the bulk background fields are known. Derivative corrections were studied in [11] (using the partition function method), in [12] (using boundary conformal field theory) and [13] (using the Seiberg-Witten map). Modulo field redefinitions, it was shown that there are no two derivative corrections and a proposal for the four derivative corrections through all orders in α' was made [12]. Supersymmetric extensions of the derivative corrections have been studied as well [14] (for some other supersymmetry inspired considerations see e.g. [15]).

For $n > 1$, the situation is more involved. Using the symmetrized trace description, a non-abelian generalization of the Born-Infeld has been proposed [16]. However, this can not be the full answer (as was in fact stressed in [16]), as the non-abelian Born-Infeld action defined in this way does not correctly reproduce the mass spectrum of strings stretched between intersecting branes [17, 18]. In fact, this action does not even allow for a supersymmetric extension [19]. Ignoring derivative corrections is equivalent to requiring that

the background fields are constant. This leads, because of $D_a F_{bc} = 0 \Rightarrow [F_{ab}, F_{cd}] = 0$, back to the abelian situation. So here, one has to deal with derivative corrections right from the start. Only partial results are known. Indeed, there are no $\mathcal{O}(\alpha')$ corrections and the $\mathcal{O}(\alpha'^2)$ corrections were calculated from open superstring amplitudes in [20]. Calculating higher order contributions from string scattering amplitudes is very involved[1]. As a consequence, alternative methods to construct the effective action have been developed. Requiring that certain BPS configurations solve the equations of motion [24] allowed one to calculate both the α'^3 [21], and the α'^4 [25], corrections (see also the summarizing equations in [26]). An alternative method, albeit confined to four dimensions, was developed in [27, 28] and agreed with the results in [21]. While very explicit, these results are very involved and not particularly illuminating.

This strongly suggests that one might first want to control the derivative corrections in the – much simpler – abelian case, possibly obtaining all order expressions. A direct calculation through string scattering amplitudes is very hard. Past experience with the calculation of the α'-corrections to the bulk equations of motion showed that the calculation of β-functions in the corresponding non-linear σ-model is a particularly powerful approach, [29]. When reformulated in $N = (2, 2)$ superspace, the calculation greatly simplifies, [30].

In the present paper we calculate the β-functions for an open string σ-model in the presence of a U(1) background field[2]. We perform the calculation in the $N = 2$ boundary superspace developed in [32] where many of the simplifying features discovered in [30] persist.

The paper is organized as follows. In the next section 2 we set up the σ-model in $N = 2$ boundary superspace. This is followed by an analysis of the calculation to be performed and a derivation of the necessary ingredients such as the superspace propagators. In section 4 we focus on the β-functions at one and two loops. These calculations are reproduced in detail in appendix B. Section 5 turns to the three loop calculation. Again we refer the reader interested in technical details to appendix C. We end with our conclusions. Conventions and a useful diagrammatic representation are developed in appendix A.

2. $N = 2$ non-linear sigma model with boundaries in $N = 2$ "boundary" superspace

In this section we present the action for an open string σ-model in a U(1) background. The treatment is greatly simplified if we formulate the model in $N = 2$ boundary superspace. Indeed, the whole U(1) structure turns out to be characterized by a single scalar potential V. The model at hand is a special case of the general setup developed in [32].

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[1]However note that the α'^3 results of [21] were verified by a string calculation, [22]. The results of [23] show that at higher order even the impossible might be possible!

[2]Note that β-function calculations (through two loops) were used to gain insight in the bulk/brane couplings [31].

We introduce chiral fields, Z^α, and anti-chiral fields , $Z^{\bar\alpha}$, $\alpha \in \{1, \ldots m\}$, satisfying the constraints[3],

$$\bar{D}Z^\alpha = DZ^{\bar\alpha} = 0. \tag{2.1}$$

In addition we need a set of fermionic constrained fields Ψ^α and $\Psi^{\bar\alpha}$, which satisfy,

$$\bar{D}\Psi^\alpha = \partial_\sigma Z^\alpha, \qquad D\Psi^{\bar\alpha} = \partial_\sigma Z^{\bar\alpha}. \tag{2.2}$$

The action, $\mathcal{S}$, consists of a free bulk term, $\mathcal{S}_0$, and a boundary interaction term, $\mathcal{S}_{int}$,

$$\mathcal{S} = \mathcal{S}_0 + \mathcal{S}_{int}, \tag{2.3}$$

where,

$$\mathcal{S}_0 = \int d^2\sigma d^2\theta \left(g_{\alpha\bar\beta}\, DZ^\alpha \bar{D}Z^{\bar\beta} + g_{\alpha\bar\beta}\, \Psi^\alpha \Psi^{\bar\beta} \right),$$
$$\mathcal{S}_{int} = -\int d\tau d^2\theta\, V\left(Z, \bar{Z}\right), \tag{2.4}$$

where the potential V in $\mathcal{S}_{int}$ is at this point an arbitrary real function of the chiral and the anti-chiral superfields. In these equations, we rescaled Z (and Ψ) by a factor $\sqrt{2\pi\alpha'}$ such as to make it dimensionless (of dimension 1/2 resp.). We choose Neumann boundary conditions,

$$\Psi^\alpha\Big|_{\text{boundary}} = \Psi^{\bar\alpha}\Big|_{\text{boundary}} = 0. \tag{2.5}$$

The boundary term gives the coupling to a U(1) background field. The magnetic fields, appropriately rescaled by a factor $2\pi\alpha'$, are obtained from the potential,

$$F_{\alpha\bar\beta} = iV_{\alpha\bar\beta}$$
$$F_{\alpha\beta} = F_{\bar\alpha\bar\beta} = 0, \tag{2.6}$$

where here and in the next we use the notation,

$$V_{\alpha_1\cdots\alpha_m\bar\beta_1\cdots\bar\beta_n} \equiv \partial_{\alpha_1}\cdots\partial_{\alpha_m}\partial_{\bar\beta_1}\cdots\partial_{\bar\beta_n}V. \tag{2.7}$$

Whenever needed we take the potentials to be,

$$A_\alpha = -\frac{i}{2}\partial_\alpha V, \qquad A_{\bar\alpha} = \frac{i}{2}\partial_{\bar\alpha}V. \tag{2.8}$$

We will treat the potential using a background field expansion: we take the superfields to be the sum of some solution of the equations of motion with the quantum fluctuations. Expanding the potential around the classical configuration, we get the interactions,

$$\mathcal{S}_{int} = -\int d\tau d^2\theta \left(\frac{1}{2}V_{\alpha\beta}Z^\alpha Z^\beta + \frac{1}{2}V_{\bar\alpha\bar\beta}Z^{\bar\alpha}Z^{\bar\beta} + V_{\alpha\bar\beta}Z^\alpha Z^{\bar\beta} + \right.$$
$$\left. \frac{1}{3!}V_{\alpha\beta\gamma}Z^\alpha Z^\beta Z^\gamma + \frac{1}{3!}V_{\bar\alpha\bar\beta\bar\gamma}Z^{\bar\alpha}Z^{\bar\beta}Z^{\bar\gamma} + \frac{1}{2}V_{\alpha\beta\bar\gamma}Z^\alpha Z^\beta Z^{\bar\gamma} + \frac{1}{2}V_{\bar\alpha\bar\beta\gamma}Z^{\bar\alpha}Z^{\bar\beta}Z^\gamma + \cdots \right), \tag{2.9}$$

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[3]Our conventions are given in appendix A.

where the vertices are functions of the background fields and the quantum fluctuations are denoted by Z. The terms linear in the fluctuations vanish because the background fields solve the equations of motion.

Before turning to the actual calculations, we will first analyze what to expect. The bare potential V_{bare} will be of the form,

$$V_{bare} = V + \sum_{r \geq 1} V_{(r)} \lambda^r, \tag{2.10}$$

with $\lambda = (\ln \Lambda)/\pi$ and $\Lambda = M/m$ with M the UV cut-off and m the IR regulator. Making the loop expansion explicit we get

$$V_{(r)} = \sum_{s \geq r} V_{(r,s)} \hbar^s. \tag{2.11}$$

Using eq. (2.8), one finds that the β-functions for the gauge potentials are then given by,

$$\beta_\alpha = \frac{i}{2\pi} \partial_\alpha V_{(1)}, \qquad \beta_{\bar{\alpha}} = -\frac{i}{2\pi} \partial_{\bar{\alpha}} V_{(1)}. \tag{2.12}$$

The renormalization group recursively expresses $V_{(r)}$, $r \geq 2$, in terms of $V_{(1)}$,

$$V_{(r+1)}(A) = -\frac{\pi}{r+1} V_{(r)}(A + \beta)\Big|_{\text{part linear in } \beta}. \tag{2.13}$$

This fact provides a strong consistency check on the loop calculations.

3. Propagators, vertices and superspace technology

In order to calculate the propagator we introduce unconstrained sources J, $\bar{J}$, Ω and $\bar{\Omega}$. Adding them to the free action,

$$\mathcal{S} = \int d^2\sigma d^2\theta \left(g_{\alpha\bar{\beta}} DZ^\alpha D Z^{\bar{\beta}} + g_{\alpha\bar{\beta}} \Psi^\alpha \Psi^{\bar{\beta}} + J_\alpha Z^\alpha + \bar{J}_{\bar{\alpha}} Z^{\bar{\alpha}} + \Omega_\alpha \Psi^\alpha + \bar{\Omega}_{\bar{\alpha}} \Psi^{\bar{\alpha}} \right), \tag{3.1}$$

we get, upon completing the squares, the propagators,

$$\langle Z^\alpha(1) Z^{\bar{\beta}}(2) \rangle = -g^{\alpha\bar{\beta}} \frac{D_2 \bar{D}_2}{\Box_2} \delta^{(4)}(1-2),$$

$$\langle \Psi^\alpha(1) \Psi^{\bar{\beta}}(2) \rangle = -g^{\alpha\bar{\beta}} \left(1 - \bar{D}_2 D_2 \frac{\partial_{\tau_2}}{\Box_2} \right) \delta^{(4)}(1-2),$$

$$\langle Z^\alpha(1) \Psi^{\bar{\beta}}(2) \rangle = -g^{\alpha\bar{\beta}} \bar{D}_2 \frac{\partial_{\sigma_2}}{\Box_2} \delta^{(4)}(1-2),$$

$$\langle \Psi^\alpha(1) Z^{\bar{\beta}}(2) \rangle = g^{\alpha\bar{\beta}} D_2 \frac{\partial_{\sigma_2}}{\Box_2} \delta^{(4)}(1-2), \tag{3.2}$$

which satisfy the boundary conditions eq. (2.5). As a consistency check, one verifies that the propagators are compatible with the constraints, eqs. (2.1) and (2.2).

For the calculation at hand, we only need the first of the propagators in eq. (3.2) evaluated at the boundary. It becomes,

$$
\mathbb{D}^{\alpha\bar{\beta}}(1-2) \equiv \langle Z^{\alpha}(1)Z^{\bar{\beta}}(2)\rangle\Big|_{\text{boundary}}
$$
$$
= g^{\alpha\bar{\beta}} D_2 \bar{D}_2 \left(\frac{1}{\pi} \int_m^M \frac{dp}{p} \cos\left(p\left(\tau_1 - \tau_2\right)\right) \delta^{(2)}(\theta_1 - \theta_2) \right)
$$
$$
= -g^{\alpha\bar{\beta}} \bar{D}_1 D_1 \left(\frac{1}{\pi} \int_m^M \frac{dp}{p} \cos\left(p\left(\tau_1 - \tau_2\right)\right) \delta^{(2)}(\theta_1 - \theta_2) \right), \qquad (3.3)
$$

where the IR regulator was denoted by m and the UV cut-off by M.

$$
\bar{\alpha} \; \longleftarrow\!\!\Vert\!\longrightarrow \; \bar{\beta}
$$

Figure 1: A diagram containing vertices with only holomorphic or only anti-holomorphic indices will be UV finite.

When calculating the β-functions, we are solely interested in the UV divergences which we will treat using minimal subtraction. The actual calculation is further simplified by several observations made in [30] which carry over to the present case. One verifies that all UV divergences are logarithmic. As the interactions are fully characterized by a dimensionless potential V, the counterterms, V_{ct}, will be dimensionless as well. This immediately implies that, as long as we are only interested in the UV divergent part of the diagrams, the interaction vertices in eq. (2.9) (which are in fact functions of the background fields) can effectively be treated as constants. A corrolary on this is that vertices with only holomorphic (or only anti-holomorphic) indices will never contribute to the UV divergences. Indeed, consider e.g. the diagram in figure (1). It gives rise to a contribution of the form,

$$
\frac{1}{2} \int d\tau_3 d^2\theta_3 V_{\gamma\delta} \mathbb{D}^{\gamma\bar{\alpha}}(3-1) \mathbb{D}^{\delta\bar{\beta}}(3-2), \qquad (3.4)
$$

which, upon partially integrating a fermionic derivative in one of the propagators, can be seen to only contribute to the UV finite part of the diagram. Exactly the same reasoning can be made with the effective propagators appearing later on. The previous also shows that the loop expansion is an expansion in the number of derivatives. Indeed the n-loop contribution to the β-functions will give rise to $2n-2$ derivatives acting on the fieldstrenghts.

Using the previous observations, one calculates the relevant effective tree level propagator. It is diagramatically shown in figure
Both even and odd numbers of vertices contribute and the final expression reads as,

$$
\mathcal{D}^{\alpha\bar{\beta}}(1-2) = \mathbb{D}^{\alpha\bar{\beta}}_+(1-2) + \mathbb{D}^{\alpha\bar{\beta}}_-(1-2), \qquad (3.5)
$$

with,

$$
\mathbb{D}^{\alpha\bar{\beta}}_\pm(1-2) = h^{\alpha\bar{\beta}}_\pm \mathbb{D}_\pm(1-2) = h^{\alpha\bar{\beta}}_\pm D_2 \bar{D}_2 \left(\delta^{(2)}(\theta_1 - \theta_2)\Delta_\pm(1-2) \right), \qquad (3.6)
$$

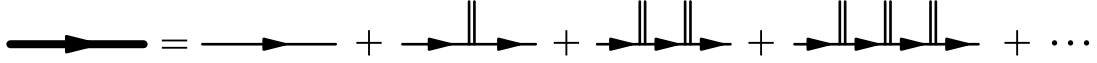**Figure 2:** The diagrammatic expansion of the effective propagator.

where $h_\pm^{\alpha\bar\beta}$ are the inverses of $h_{\alpha\bar\beta}^\pm \equiv g_{\alpha\bar\beta} \pm F_{\alpha\bar\beta}$, see eqs. (A.1) and (A.2), and

$$\Delta_\pm(\tau) = \frac{1}{2\pi} \int_m^M \frac{dp}{p}\, e^{\mp i\, p\, \tau}. \tag{3.7}$$

Note that $\Delta_+(-\tau) = \Delta_-(\tau)$.

4. One and two loop contributions

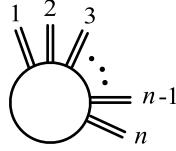As a warming up exercise, we calculate the one and two loop contributions in some detail.

Figure 3: A one loop diagram with n vertices. The propagators are the free ones, eq. (3.3).

Performing the D-algebra, one finds that a one loop diagram with $2n$ vertices is given by,

$$\frac{1}{2n\, i} \int d\tau_1 d^2\theta_1 d\tau_2 d^2\theta_2 \left(F^{2n}\right)_{\alpha\bar\beta} g^{\alpha\bar\beta}\big(\Delta_+(\tau_1 - \tau_2) - \Delta_-(\tau_1 - \tau_2)\big)\delta^{(3)}(1-2), \tag{4.1}$$

where we used eq. (2.6) and introduced the notation,

$$\left(F^m\right)_{\alpha\bar\beta} \equiv F_{\alpha\bar\gamma_1} g^{\bar\gamma_1\delta_1} F_{\delta_1\bar\gamma_2} g^{\bar\gamma_2\delta_2} F_{\delta_2\bar\gamma_3} \cdots g^{\bar\gamma_{m-1}\delta_{m-1}} F_{\delta_{m-1}\bar\beta}. \tag{4.2}$$

Because $\Delta_+(\tau_1 - \tau_2) - \Delta_-(\tau_1 - \tau_2)$ is an odd function, this vanishes. Turning to a one loop diagram with $2n+1$ vertices, one finds an ultra-violet divergence of the form,

$$-i\frac{\lambda}{2n+1} \int d\tau d^2\theta\, g^{\alpha\bar\beta} \left(F^{2n+1}\right)_{\alpha\bar\beta}, \tag{4.3}$$

where,

$$\lambda \equiv \frac{1}{\pi} \ln\left(\frac{M}{m}\right). \tag{4.4}$$

When summing over all loops, we get the total divergent contribution

at one loop,

$$-i\lambda \int d\tau d^2\theta \, g^{\alpha\bar{\beta}} \left(\text{arcth} \, F\right)_{\alpha\bar{\beta}}, \tag{4.5}$$

where,

$$\left(\text{arcth} F\right)_{\alpha\bar{\beta}} \equiv F_{\alpha\bar{\beta}} + \frac{1}{3} F_{\alpha\bar{\gamma}} \, g^{\bar{\gamma}\delta} \, F_{\delta\bar{\epsilon}} \, g^{\bar{\epsilon}\eta} \, F_{\eta\bar{\beta}} + \cdots \tag{4.6}$$

Using minimal subtraction, this gives us the bare potential through order $\hbar$,

$$V_{bare} = V + i\lambda \, g^{\alpha\bar{\beta}} \left(\text{arcth} \, F\right)_{\alpha\bar{\beta}}. \tag{4.7}$$

Let us first discuss the UV properties of the model at one loop. One reads from eq. (4.5) that the non-linear σ-model defined by eqs. (2.3) and (2.4) is UV finite at one loop provided,

$$g^{\alpha\bar{\beta}} \left(\text{arcth} \, F\right)_{\alpha\bar{\beta}} = 0, \tag{4.8}$$

holds. This equation appeared elsewhere as well. Indeed in [33], the well known four dimensional instanton equations were generalized to higher dimensions. The generic class of such configurations are now known as stable holomorphic bundles [34]. They are easily characterized if we limit ourselves to a flat (even dimensional and Euclidean) spacetime. Passing to complex coordinates, one finds using the Bianchi identities that magnetic fields which satisfy the linear relations,

$$F_{\alpha\beta} = F_{\bar{\alpha}\bar{\beta}} = 0,$$
$$g^{\alpha\bar{\beta}} F_{\alpha\bar{\beta}} = 0, \tag{4.9}$$

automatically solve the Yang-Mills equations of motion. The first line is the holomorphicity condition, the second is the stability condition. Later on, these equations were (at least for sufficiently low dimensions) recognized as BPS equations for supersymmetric Yang-Mills theories (for a detailed discussion, see e.g. [35] and references therein). The starting point of [24] was the most general deformation of the Maxwell action involving higher powers of the fieldstrength but excluding derivatives acting on the fieldstrengths. Requiring that solutions of the type given in eq. (4.9) still exist, uniquely fixes the deformation of the Maxwell action: it is the Born-Infeld action. The holomorphicity condition in eq. (4.9) remains unchanged while the stability condition gets deformed to precisely eq. (4.8). Concluding, we see that requiring the model defined by eqs. (2.3) and (2.4) to be UV finite (at one loop) can be reinterpreted as requiring that the holomorphic bundle satisfies the deformed stability condition as well.

Turning back to the β-functions, we find using eq. (2.12), that they vanish provided,

$$g^{\beta\bar{\gamma}} \partial_\alpha \left(\text{arcth} F\right)_{\beta\bar{\gamma}} = 0, \tag{4.10}$$

holds. eq. (4.10) arises as the equation of motion for the Born-Infeld action. Indeed, varying the Born-Infeld action $\mathcal{S}_{BI}$,

$$\mathcal{S}_{BI} = -\int d^{2m}X \, \sqrt{\det h_+}, \tag{4.11}$$

yields,

$$\delta S_{BI} = \int d^{2m}X \sqrt{\det h_+}\, \delta A_a \mathcal{G}^{ab} \mathcal{G}^{cd} \partial_c F_{db}, \tag{4.12}$$

where $\mathcal{G}^{ab}$ is defined in eq. (A.3). Passing to complex coordinates and implementing the holomorphicity, $F_{\alpha\beta} = F_{\bar{\alpha}\bar{\beta}} = 0$, we get from eq. (4.12) the equations of motion,

$$\mathcal{G}^{\beta\bar{\gamma}} \partial_\beta F_{\bar{\gamma}\alpha} = 0, \tag{4.13}$$

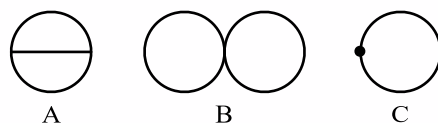which upon using the Bianchi identities reduces to eq. (4.10).

Figure 4: The two 2-loop diagrams and the contribution arising from the one loop counter term. The effective propagator, eq. (3.5), is used.

We now turn to the two loop contributions. Technical details of the calculation can be found in appendix B. Diagram A in figure (4) results in,

$$-\frac{\lambda^2}{2} \int d\tau d^2\theta\, V_{\bar{\beta}_1 \alpha_2 \alpha_3} V_{\alpha_1 \bar{\beta}_2 \bar{\beta}_3} \left(\mathcal{G}^{\alpha_1 \bar{\beta}_1} \mathcal{G}^{\alpha_2 \bar{\beta}_2} \mathcal{B}^{\alpha_3 \bar{\beta}_3} + \mathcal{B}^{\alpha_1 \bar{\beta}_1} \mathcal{G}^{\alpha_2 \bar{\beta}_2} \mathcal{G}^{\alpha_3 \bar{\beta}_3} \right), \tag{4.14}$$

where,

$$\mathcal{B}^{ab} \equiv \frac{1}{2\,i} \left(h_+^{ab} - h_-^{ab} \right). \tag{4.15}$$

Diagram B gives,

$$\frac{\lambda^2}{2} \int d\tau d^2\theta\, \mathcal{G}^{\alpha\bar{\beta}} \mathcal{G}^{\gamma\bar{\delta}} V_{\alpha\bar{\beta}\gamma\bar{\delta}}. \tag{4.16}$$

Adding the two gives,

$$-i\frac{\lambda^2}{2} \int d\tau d^2\theta\, \mathcal{G}^{\alpha\bar{\beta}} \partial_\alpha \partial_{\bar{\beta}} \left(g^{\gamma\bar{\delta}} (\operatorname{arcth} F)_{\gamma\bar{\delta}} \right). \tag{4.17}$$

Finally, the subdivergence (diagram C), which arises from the one loop counter term, has the same structure and it is given by,

$$i\,\lambda^2 \int d\tau d^2\theta\, \mathcal{G}^{\alpha\bar{\beta}} \partial_\alpha \partial_{\bar{\beta}} \left(g^{\gamma\bar{\delta}} (\operatorname{arcth} F)_{\gamma\bar{\delta}} \right). \tag{4.18}$$

Adding these contributions gives us the bare potential through two loops.

$$V_{bare} = V + \lambda\, V_{(1)} + \lambda^2\, V_{(2)}, \tag{4.19}$$

where

$$V_{(1)} = ig^{\alpha\bar{\beta}} \left(\text{arcth}\, F\right)_{\alpha\bar{\beta}}$$

$$V_{(2)} = -\frac{i}{2} \mathcal{G}^{\alpha\bar{\beta}} \partial_\alpha \partial_{\bar{\beta}} \left(g^{\gamma\bar{\delta}} \left(\text{arcth}\, F\right)_{\gamma\bar{\delta}} \right) \tag{4.20}$$

$$= -\frac{1}{2} \bigcirc S \,.$$

In the last line we used the diagrammatic notation introduced in appendix A. One easily verifies that the two loop $\mathcal{O}(\lambda^2)$ counterterm agrees with the renormalization group result, eq. (2.13).

As the bare potential does not get an order λ correction at two loops, the β-function will not be modified at this order and as a consequence, the Born-Infeld action receives no two derivative corrections. This result agrees with [11].
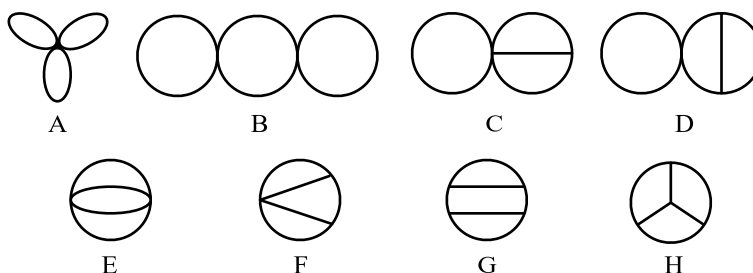
5. Three loop result

Figure 5: All the 3-loop diagrams. The effective propagator, eq. (3.5), is used.

The 3-loop diagrams are shown in figure (5). As is explained in more detail in appendix C, the three loop diagrams give rise to terms proportional to λ^3 as well as terms linear in λ. It is a non-trivial check on the calculation that, with the contributions from the one and two loop counterterms taken into account, there are no terms quadratic in λ present at this order. This must be the case, since there was no contribution to the β-function at the two loop level. The λ^3 terms can be expressed quite concisely as

$$V_{(3)} = -\frac{1}{3} \mathcal{G}^{\alpha\bar{\beta}} \partial_\alpha \partial_{\bar{\beta}} V_{(2)} - \frac{i}{12} \left(h_+^{\alpha\bar{\beta}} h_+^{\gamma\bar{\delta}} - h_-^{\alpha\bar{\beta}} h_-^{\gamma\bar{\delta}} \right) \partial_\alpha \partial_{\bar{\delta}} V_{(1)} \partial_\gamma \partial_{\bar{\beta}} V_{(1)}$$

$$= -\frac{1}{3} \bigcirc (2) - \frac{i}{12} \left((1) \bigcirc (1) - (1) \bigcirc (1) \right) . \tag{5.1}$$

A very strong check on the calculation is the fact that this precisely agrees with the renormalization group equations in eq. (2.13)! In the second line we again used the diagrammatic

notation explained in appendix (A). Adding the terms linear in λ to the one loop result, we find up to terms containing four derivatives of the fieldstrength

$$V_{(1)} = ig^{\alpha\bar\beta}\left(\text{arcth}\,\tilde F\right)_{\alpha\bar\beta} - \frac{1}{48}S_{ab\alpha\bar\beta}S_{cd\gamma\bar\delta}\,h_+^{bc}h_+^{da}\left(\mathcal{G}^{\alpha\bar\delta}\mathcal{B}^{\gamma\bar\beta} + \mathcal{B}^{\alpha\bar\delta}\mathcal{G}^{\gamma\bar\beta}\right) + \mathcal{K}V_{(1,1)}\,. \quad (5.2)$$

Here $\tilde F$ is the fieldstrength associated with the gauge potential $\tilde A$, related to the original A by the field redefinition

$$\tilde A_\alpha = A_\alpha + \frac{1}{24}\partial_\alpha\left(F_{\beta_1\bar\gamma_2,\bar\gamma_3}F_{\beta_2\bar\gamma_1,\beta_3}h_+^{\beta_1\bar\gamma_1}h_+^{\beta_2\bar\gamma_2}\mathcal{G}^{\beta_3\bar\gamma_3}\right)$$

$$= A_\alpha + \frac{1}{24}\partial_\alpha\,\text{[diagram]};$$

(5.3)

$$\tilde A_{\bar\alpha} = A_{\bar\alpha} + \frac{1}{24}\partial_{\bar\alpha}\left(F_{\beta_1\bar\gamma_2,\bar\gamma_3}F_{\beta_2\bar\gamma_1,\beta_3}h_-^{\beta_1\bar\gamma_1}h_-^{\beta_2\bar\gamma_2}\mathcal{G}^{\beta_3\bar\gamma_3}\right)$$

$$= A_{\bar\alpha} + \frac{1}{24}\partial_{\bar\alpha}\,\text{[diagram]}.$$

Note that we can omit the tilde on the fields appearing in the second term in eq. (5.2), because this has only an effect on the terms which are higher order in $\hbar$. We also introduced (see [12])

$$S_{abcd} = \partial_a\partial_b F_{cd} + h_+^{ef}\,\partial_a F_{ce}\,\partial_b F_{df} - h_+^{ef}\,\partial_a F_{de}\,\partial_b F_{cf}. \quad (5.4)$$

The use of Latin indices indicates a summation over real coordinates, where the use of Greek indices, as usual, refers to a summation over complexified holomorphic or anti-holomorphic coordinates. Finally, in the last term of eq. (5.2), $\mathcal{K}$ is a derivative operator acting on the one loop part of $V_{(1)}$ (see eq. (2.11) for notation). This term can be written in a fairly transparent way by again making use of our diagrammatic notation:

$$\mathcal{K}V_{(1,1)} = \frac{1}{24}\Big(\text{[diagram]} - \text{[diagram]} + \text{[diagram]} - \text{[diagram]}$$

$$+ \text{[diagram]} + \text{[diagram]} + \text{[diagram]} + \text{[diagram]}\Big)$$

$$+ \frac{1}{48}\Big(\text{[diagram]} - \text{[diagram]} - \text{[diagram]} + \text{[diagram]}\Big)$$

(5.5)

Since up to this order, we can put $V_{(1,1)}$ to zero and, in general, a field redefinition has no physical consequences, we arrive at the following correction to the stability condition for holomorphic vector bundles,

$$g^{\alpha\bar\beta}\left(\text{arcth}\,F\right)_{\alpha\bar\beta} + \frac{i}{48}S_{ab\alpha\bar\beta}S_{cd\gamma\bar\delta}\,h_+^{bc}h_+^{da}\left(\mathcal{G}^{\alpha\bar\delta}\mathcal{B}^{\gamma\bar\beta} + \mathcal{B}^{\alpha\bar\delta}\mathcal{G}^{\gamma\bar\beta}\right) = 0. \quad (5.6)$$

This is exactly the stability condition which follows from the action presented in [12] (see [35]):

$$
\mathcal{S} = -\tau_9 \int d^{10}x \, \sqrt{h_+} \Big[1 + \frac{1}{96} \Big(\frac{1}{2} h_+^{ab} h_+^{cd} S_{bc} S_{da}
$$
$$
- h_+^{c_2 a_1} h_+^{a_2 c_1} h_+^{d_2 b_1} h_+^{b_2 d_1} S_{a_1 a_2 b_1 b_2} S_{c_1 c_2 d_1 d_2} \Big) \Big], \tag{5.7}
$$

where

$$
S_{ab} = h_+^{cd} S_{abcd}. \tag{5.8}
$$

6. Conclusions and outlook

In the present paper we calculated the β-functions through three loops for an open string σ-model in the presence of U(1) background. Requiring them to vanish is then reinterpreted as the string equations of motion for the background. Upon integration this yields the low energy effective action. Doing the calculation in $N = 2$ boundary superspace significantly simplified the calculation. The one loop contribution gives the effective action to all orders in α' in the limit of a constant fieldstrength. The result is the well known Born-Infeld action. The absence of a two loop contribution to the β-function shows the absence of two derivative terms in the action. Finally the three loop contribution gives the four derivative terms in the effective action to all orders in α'. Modulo a field redefinition we find complete agreement with the proposal made in [12].

By doing the calculation in $N = 2$ superspace, we get a nice geometric characterization of UV finiteness of the non-linear σ-model: UV finiteness is guaranteed provided that the background is a deformed stable holomorphic bundle.

An immediate question is whether the present program can be pushed to higher orders. Of course, already at four loops this procedure becomes extremely cumbersome. There might however be general arguments that lead to considerable simplifications. First of all, consider a diagram with an external loop, which will in general look like diagram A of figure (6), where the bigger circle with shaded area can be any diagram[4]. It is not very difficult to show that diagram B of figure (6), resulting from replacing that external loop with the one loop counterterm, will contain a term which exactly cancels the original diagram. As a consequence, diagrams with (one or more) external loops cannot contribute to the β-function. This is of course part of the bigger renormalization group picture. Diagrams which factorize will never contribute to the β-function, because the divergences encountered in the corresponding loop-integrals are already accounted for at a lower-loop level.

More important simplifications might arise from an interesting observation made in [12]. There Niclas Wyllard noted that S_{abcd} which was introduced in eq. (5.4) can be viewed as the curvature tensor for a non-symmetric connection. Once this is better understood, this could lead to a method giving results to all order in the derivatives. Indeed the leading

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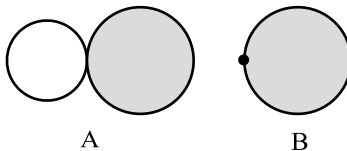[4]For some examples, see diagrams A - D of figure (5)

Figure 6: A general diagram with external loop (A) and that same diagram with the external loop replaced by the one loop counterterm (B).

contribution to the β-functions comes from the n-loop "onion" diagram shown in figure (7) which can be explicitly calculated in a reasonably straightforward way. The remainder of the β-function should then follow as some sort of covariantization. This point of view is presently under investigation [36].

Figure 7: An n-loop "onion" diagram.

Finally, a natural question which arises here is whether the present method extends to the non-abelian case. In that case the coupling to the gauge fields involves the introduction of a Wilson line. The path-ordering can be undone through the introduction of auxiliary fields, [37], and a first exploration was performed in [5]. However, before the present analysis can be done for the non-abelian case, one needs to extend the superspace formulation in [32] such as to include Wilson loops and the auxiliary formulation of [37]. This would certainly lead to significant information on non-abelian deformed stable holomorphic bundles. We leave this interesting question to future investigation.

Acknowledgments

A. Conventions, notations and identities

Both the worldsheet as well as the target space carry a flat euclidean metric. The target space has dimension $d = 2n$, $n \in \mathbb{N}$ and we use Latin indices to denote real and greek indices for complex coordinates. We write the metric as g_{ab} and we have that $g_{ab} = \delta_{ab}$. The magnetic fields, F_{ab}, are incorporated in the open string metric(s), $h_{ab}^{\pm}$,

$$h_{ab}^{\pm} \equiv g_{ab} \pm F_{ab}, \tag{A.1}$$

and we obviously have that $h_{ab}^+ = h_{ba}^-$. The inverse, $h_\pm^{ab}$, is defined by,

$$h_+^{ac} h_{cb}^+ = h_-^{ac} h_{cb}^- = \delta_b^a. \tag{A.2}$$

In addition we define,

$$\mathcal{G}^{ab} \equiv \frac{1}{2} \left(h_+^{ab} + h_-^{ab} \right). \tag{A.3}$$

In complex coordinates, the metric of the target space is given by,

$$g_{\alpha\beta} = g_{\bar{\alpha}\bar{\beta}} = 0, \qquad g_{\alpha\bar{\beta}} = \frac{1}{2}\,\delta_{\alpha\bar{\beta}}. \tag{A.4}$$

The $N = 2$ supersymmetry requires the U(1) bundle to be holomorphic,

$$F_{\alpha\beta} = F_{\bar{\alpha}\bar{\beta}} = 0, \tag{A.5}$$

which implies that,

$$h_{\alpha\beta}^\pm = h_{\bar{\alpha}\bar{\beta}}^\pm = 0. \tag{A.6}$$

For much of the superspace techniques, we refer to the "bible": [38]! The $N = 2$ boundary superspace is parameterized by two bosonic coordinates $\tau \in \mathbb{R}$, and $\sigma \in \mathbb{R}$, $\sigma \geq 0$, and two fermionic coordinates θ and $\bar{\theta}$. The fermionic derivatives are defined by,

$$D\theta = \bar{D}\bar{\theta} = 1, \qquad D\bar{\theta} = \bar{D}\theta = 0, \tag{A.7}$$

and,

$$D^2 = \bar{D}^2 = 0, \qquad \{D, \bar{D}\} = \partial_\tau. \tag{A.8}$$

The superspace integration measure is defined by,

$$\int d^2\theta \, \bar{\theta}\theta = 1. \tag{A.9}$$

Some definitions involving δ-functions,

$$\begin{aligned}
\delta^{(2)}(\theta_1 - \theta_2) &\equiv (\bar{\theta}_1 - \bar{\theta}_2)(\theta_1 - \theta_2), \\
\delta^{(3)}(1 - 2) &\equiv \delta(\tau_1 - \tau_2)\delta^{(2)}(\theta_1 - \theta_2), \\
\delta^{(4)}(1 - 2) &\equiv \delta^{(2)}(\sigma_1 - \sigma_2)\delta^{(2)}(\theta_1 - \theta_2),
\end{aligned} \tag{A.10}$$

and some useful identities,

$$\begin{aligned}
&\delta^{(2)}(\theta_1 - \theta_2)\delta^{(2)}(\theta_2 - \theta_1) = 0, \\
&\delta^{(2)}(\theta_1 - \theta_2)D_a\left(\delta^{(2)}(\theta_2 - \theta_1)f(\tau_2 - \tau_1)\right) = 0, \\
&\delta^{(2)}(\theta_1 - \theta_2)\bar{D}_a\left(\delta^{(2)}(\theta_2 - \theta_1)f(\tau_2 - \tau_1)\right) = 0, \\
&\delta^{(2)}(\theta_1 - \theta_2)\left(D_a\bar{D}_a\delta^{(2)}(\theta_2 - \theta_1)f(\tau_2 - \tau_1)\right) = \\
&-\delta^{(2)}(\theta_1 - \theta_2)\left(\bar{D}_a D_a\delta^{(2)}(\theta_2 - \theta_1)f(\tau_2 - \tau_1)\right) = \delta^{(2)}(\theta_1 - \theta_2)f(\tau_2 - \tau_1),
\end{aligned} \tag{A.11}$$

where the subindex a is 1 or 2 and where we did not sum over repeated subindices a.

In keeping track of all contributions to the bare potential and checking the renormalization group equations it proved to be very helpful to introduce a diagrammatic notation for the index structure of the different contributions. One of the most attractive features of this notation is that the diagram of the index structure is, as will become clear, exactly the same as the diagram of the full contribution it corresponds to. Of course, this same feature is also a possible cause for confusion. We hope however that the precise meaning of the diagram will always be clear from the context. When diagrams are used several times in the same formula, not necessarily always with the same meaning, the part that indicates the index structure only (without the momentum integral and symmetry factor), will appear in between brackets. (For an example of this, see eq. (B.13) below.) More concretely, we introduce the notations

$$h_{\pm}^{\alpha\bar{\beta}} = \alpha \underset{\pm}{\longrightarrow} \bar{\beta}, \tag{A.12}$$

$$\mathcal{G}^{\alpha\bar{\beta}} = \frac{1}{2}\left(h_{+}^{\alpha\bar{\beta}} + h_{-}^{\alpha\bar{\beta}}\right) = \alpha \underset{\mathrm{S}}{\longrightarrow} \bar{\beta}, \tag{A.13}$$

$$\partial_{\gamma}F_{\alpha\bar{\beta}} = \partial_{\alpha}F_{\gamma\bar{\beta}} = \tag{A.14}$$

and

$$\partial_{\bar{\gamma}}F_{\alpha\bar{\beta}} = \partial_{\bar{\beta}}F_{\alpha\bar{\gamma}} = \tag{A.15}$$

Here it should be clear that $\alpha\blacktriangleleft\!\!\!-$ and $-\!\!\!\blacktriangleleft\bar{\beta}$ represent derivatives on the field strength or potential and should not be confused with the object defined in eq. (A.12) corresponding to the propagator.

It is very easy to compute derivatives of different expressions by using (for example)

$$\partial_{\bar{\gamma}}h_{\pm}^{\alpha\bar{\beta}} = \pm h_{\pm}^{\alpha\bar{\delta}}h_{\pm}^{\varepsilon\bar{\beta}}\,\partial_{\bar{\gamma}}F_{\varepsilon\bar{\delta}}. \tag{A.16}$$

In diagrammatic form this becomes

$$\partial_{\bar{\gamma}}\left(\alpha \underset{\pm}{\longrightarrow} \bar{\beta}\right) = \pm \tag{A.17}$$

The same formula evidently also holds for derivatives with respect to holomorphic coordinates. Also very useful is the following identity:

$$\partial_{\bar{\beta}}V_{(1,1)} = \partial_{\bar{\beta}}\left(i\,g^{\gamma\bar{\delta}}\,(\operatorname{arcth}F)_{\gamma\bar{\delta}}\right) = i\mathcal{G}^{\gamma\bar{\delta}}\partial_{\bar{\beta}}F_{\gamma\bar{\delta}} = i \tag{A.18}$$

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Applying yet another derivative results in

$$\partial_\alpha \partial_{\bar\beta} V_1 = i \left[\begin{array}{c} \text{S} \\ \bigcirc \\ \alpha \qquad \bar\beta \end{array} + \frac{1}{2} \left(\begin{array}{c} + \\ \alpha \bigcirc \bar\beta \\ + \end{array} - \begin{array}{c} - \\ \alpha \bigcirc \bar\beta \\ - \end{array} \right) \right] \quad \text{(A.19)}$$

B. Explicit two loop computation

As an illustration of how the calculation was done, we will now explicitly compute the two loop contribution. Let us first consider diagram A in figure (4). Of course the effective propagator, eq. (3.5), has a certain direction, so that diagram A actually represents all possible inequivalent topologies with directed lines. Since the righthand side of eq. (3.5) also consists of two terms, in the end we have to consider all possible distinct diagrams with directed lines and all possible combinations of positive and negative 'frequency' parts of the propagator (eq. (3.6)). It turns out that the only momentum integrals that have to be performed explicitly, are the ones belonging to the two diagrams in figure (8). All other integrals are either zero (or finite, and therefore irrelevant for our beta function computation), or are related to these ones by complex conjugation or partial integration.
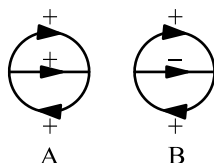

Figure 8: The basic non-trivial diagrams at two loops.

Let us start with diagram A of figure (8). The Feynman rules for this diagram give:

$$\frac{1}{2} \left\langle V_{\bar\beta_1 \alpha_2 \alpha_3}(1) V_{\alpha_1 \bar\beta_2 \bar\beta_3}(2) \mathbb{D}_+^{\alpha_1 \bar\beta_1}(2-1) \mathbb{D}_+^{\alpha_2 \bar\beta_2}(1-2) \mathbb{D}_+^{\alpha_3 \bar\beta_3}(1-2) \right\rangle_{1,2} , \quad \text{(B.1)}$$

where $\langle \rangle_1$ means integrating over τ_1, θ_1 and $\bar\theta_1$. Using eq. (3.6) and performing the D-algebra leads to

$$\frac{1}{2} \left\langle V_{\bar\beta_1 \alpha_2 \alpha_3} V_{\alpha_1 \bar\beta_2 \bar\beta_3} h_+^{\alpha_1 \bar\beta_1} h_+^{\alpha_2 \bar\beta_2} h_+^{\alpha_3 \bar\beta_3} \mathcal{I}_A \right\rangle_1 , \quad \text{(B.2)}$$

with

$$\mathcal{I}_A = \int_{-\infty}^{+\infty} d\tau \ \Delta_+(\tau) \Delta_+(\tau) \partial_\tau \Delta_-(\tau) , \quad \text{(B.3)}$$

where we made the change of variable $\tau_2 \to \tau \equiv \tau_1 - \tau_2$ and made use of $\Delta_+(-\tau) = \Delta_-(\tau)$. Inserting the explicit form of the coordinate space propagators eq. (3.7) and using the fact that

$$\partial_\tau \Delta_\pm(\tau) = \frac{1}{2\pi\tau}(e^{\mp iM\tau} - e^{\mp im\tau}) , \quad \text{(B.4)}$$

we arrive at the following expression:

$$\mathcal{I}_A = \frac{1}{(2\pi)^3} \int_m^M \frac{dp}{p} \int_m^M \frac{dq}{q} \int \frac{d\tau}{\tau} \left(e^{-i(p+q-M)\tau} - e^{-i(p+q-m)\tau} \right) \tag{B.5}$$

$$= -\frac{i\pi}{(2\pi)^3} \int_1^\Lambda \frac{dp}{p} \int_1^\Lambda \frac{dq}{q} \left[\epsilon(p+q-\Lambda) - \epsilon(p+q-1) \right] , \tag{B.6}$$

where we used the sign function $\epsilon(p) = 1$ for $p > 0$ and $\epsilon(p) = -1$ for $p < 0$. In the last line all momenta are expressed in units of m and again $\Lambda = M/m$. This integral can easily be written in terms of logarithms and dilogarithms,

$$\mathcal{I}_A = \frac{i}{(2\pi)^2} \left[\ln(\Lambda - 1)\ln\Lambda + \text{Li}_2\left(\frac{1}{\Lambda}\right) - \text{Li}_2\left(\frac{\Lambda - 1}{\Lambda}\right) \right] , \tag{B.7}$$

with (the restriction to $|x| \le 1$ is only required for the second equality)

$$\text{Li}_2(x) = -\int_0^x dz \, \frac{\ln(1-z)}{z} = \sum_{k=1}^\infty \frac{x^k}{k^2} , \quad |x| \le 1 . \tag{B.8}$$

As we are only interested in the UV divergent part of $\mathcal{I}_A$, we only need the behavior of eq. (B.7) for $\Lambda \gg 1$: [5]

$$\mathcal{I}_A = -\frac{i}{24} + \frac{i}{(2\pi)^2} \ln^2(\Lambda) + \mathcal{O}\left(\frac{1}{\Lambda}\right) , \tag{B.9}$$

where we used $\text{Li}_2(1) = \zeta(2) = \pi^2/6$, as is clear from eq. (B.8). Putting everything together, we find the following contribution from diagram A of figure (8) to eq. (4.14):

$$i \frac{\lambda^2}{8} \int d\tau d^2\theta \, V_{\bar{\beta}_1 \alpha_2 \alpha_3} V_{\alpha_1 \bar{\beta}_2 \bar{\beta}_3} h_+^{\alpha_1 \bar{\beta}_1} h_+^{\alpha_2 \bar{\beta}_2} h_+^{\alpha_3 \bar{\beta}_3} . \tag{B.10}$$

It is readily verified that this is indeed one of the terms appearing in eq. (4.14).

Turning now to diagram B of figure (8), one finds that the equivalent of $\mathcal{I}_A$ for this diagram is

$$\mathcal{I}_B = \int_{-\infty}^{+\infty} d\tau \, \Delta_+(\tau) \Delta_-(\tau) \partial_\tau \Delta_-(\tau) , \tag{B.11}$$

However, this integral at its turn can be related to $\mathcal{I}_A$ by partial integration and complex conjugation:

$$\mathcal{I}_B = -\frac{1}{2} \int_{-\infty}^{+\infty} d\tau \, \Delta_-(\tau) \Delta_-(\tau) \partial_\tau \Delta_+(\tau) = -\frac{1}{2}\mathcal{I}_A^* = \frac{1}{2}\mathcal{I}_A , \tag{B.12}$$

where in the last equality we used the fact that $\mathcal{I}_A$ is purely imaginary. Diagram B of figure (4) almost trivially leads to (4.16), so that it turns out that diagram A of figure (8)

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[5]The first term on the right hand side is irrelevant when we only consider two loop contributions. However, at the three loop level (for instance for diagrams C and D of figure (5)) this result will be multiplied by $\log(\Lambda)$, coming from the extra loop. This means that the first term will come into play, while the subleading terms of order $1/\Lambda$ and higher will continue to be irrelevant for the UV behavior.

gives the only nontrivial integral we have to compute at two loops. The total contribution from diagrams A and B in figure (4) can very nicely and suggestively be written using the diagrammatic notation for the index structure explained in the previous section:

$$\bigcirc\bigcirc + \ominus = -\frac{i}{2}\lambda^2 \left\langle \text{s}\bigcirc\bigcirc\text{s} + \frac{1}{2}\left(\oplus - \ominus\right)\right\rangle \quad \text{(B.13)}$$

Finally, the contribution from diagram C in figure (4) has to be taken into account. Using eq. (A.19) one finds

$$\bullet\bigcirc = \lambda^2 \left\langle \mathcal{G}^{\alpha\bar{\beta}}\partial_\alpha\partial_{\bar{\beta}}V_1 \right\rangle$$

$$= i\lambda^2 \left\langle \text{s}\bigcirc\bigcirc\text{s} + \frac{1}{2}\left(\oplus - \ominus\right)\right\rangle \quad \text{(B.14)}$$

Comparing this to (B.13) it is easy to understand why indeed we could write eq. (4.17) the way we did. Adding all two loop contributions, we indeed find (4.20).

While the use of this diagrammatic notation might seem a bit overdoing it at two loops, it becomes unavoidable at three loops. Of course, even when using our diagrammatic notation, calculations will become quite lengthy at that stage, so we will not show any of these here explicitly. Let us only state that heavy use of this notation was made in checking eq. (5.1) and working out the consequences of the field redefinition (5.3).

C. Outline of the three loop calculation

Before we start explaining the general procedure we used for calculating the three loop contributions, it is useful to understand some general features for any number of loops. Let v be the number of vertices in the diagram under consideration, e the number of edges (propagators) and l the number of loops. These have to satisfy the topological relation $v - e + l = 1$. It follows from the D-algebra that the number of derivatives appearing in the equivalent of eq. (B.3), which we will call d, equals $v - 1$.[6] From this we can conclude that the number of remaining momentum integrations will be $e - d = e - v + 1 = l$, which, of course, makes very good sense. More importantly, the fact that in general there will be more than one derivative appearing in the equivalent of eq. (B.3), will lead to products of multiple sign functions in the equivalent of eq. (B.6), $d = v - 1$ of them to be precise.

Now, let us focus on $l = 3$. One always ends up with having to do three momentum integrals of the type appearing in eq. (B.6) with, in general, a product of one, two or three sign functions. To this end, one separates the 3-dimensional domain of integration into smaller parts, such that on each part the product of the appearing sign functions has a definite value. From the definition of the dilogarithm, eq. (B.8), and the trilogarithm,

$$\text{Li}_3(x) = \int_0^x dz \, \frac{\text{Li}_2(z)}{z} = \sum_{k=1}^{\infty} \frac{x^k}{k^3} \,, \quad |x| \leq 1 \,, \quad \text{(C.1)}$$

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[6]This will always equal the number of τ integrations (number of vertices minus the global position of the diagram), so that the procedure will always make sense.

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it is clear that one ends up with expressions involving logarithms, dilogarithms and trilogarithms. Fortunately, since we are only interested in the behavior of these expressions for $\Lambda \gg 1$, we can always convert them to expressions involving only logarithms (up to terms of order $1/\Lambda$). This will be accomplished by repeated use of identities such as [39]:

$$\text{Li}_n(1) = \zeta(n) , \tag{C.2}$$

$$\text{Li}_2(-x) + \text{Li}_2\left(-\frac{1}{x}\right) = -\frac{\pi^2}{6} - \frac{1}{2}\ln^2(x) , \quad x > 0 \tag{C.3}$$

$$\text{Li}_2(x) + \text{Li}_2\left(\frac{1}{x}\right) = \frac{\pi^2}{3} - i\pi\ln x - \frac{1}{2}\ln^2(x) , \quad x > 1 \tag{C.4}$$

$$\text{Li}_3(-x) - \text{Li}_3\left(-\frac{1}{x}\right) = -\frac{\pi^2}{6}\ln x - \frac{1}{6}\ln^3(x) , \quad x > 0 \tag{C.5}$$

$$\text{Li}_3(x) - \text{Li}_3\left(\frac{1}{x}\right) = \frac{\pi^2}{3}\ln x - i\frac{\pi}{2}\ln^2(x) - \frac{1}{6}\ln^3(x) , \quad x > 1 . \tag{C.6}$$

The most important consequence of the form of these equations is the appearance of terms linear in $\lambda \sim \ln\Lambda$, because these lead to contributions to the β-function.

We saw in the previous section that, in the end we only needed to perform one nontrivial integral to be able to calculate the entire two loop contribution. At three loops, we would of course also like to narrow down the number of necessary integrals to perform explicitly as much as possible. The divergences of diagrams A - D of figure (5) can be computed from those at two loops (and are, as already stated, irrelevant for the β-function). To be able to compute the contributions from the other diagrams, E - H of figure (5), it turns out that we still have to perform 19 integrals in total. For completeness, we list the divergent part of these integrals below.

$$\mathcal{I}_A = \int d\tau \, \Delta_+(\tau)\Delta_+(\tau)\Delta_+(\tau)\partial_\tau\Delta_-(\tau) = \frac{i}{8}\left(\lambda^3 - \frac{\lambda}{2}\right) \tag{C.7}$$

$$\mathcal{I}_B = \int d\tau \, \Delta_+(\tau)\Delta_+(\tau)\Delta_-(\tau)\partial_\tau\Delta_-(\tau) = \frac{i}{12}\lambda^3 \tag{C.8}$$

$$\mathcal{I}_C = \int d\tau_1 d\tau_2 \, \Delta_+(1)\Delta_+(1+2)\Delta_+(2)\partial_1\Delta_-(1)\partial_2\Delta_-(2) = -\frac{1}{8}\left(\lambda^3 - \frac{\lambda}{3}\right) \tag{C.9}$$

$$\mathcal{I}_D = \int d\tau_1 d\tau_2 \, \Delta_+(1)\Delta_+(1+2)\Delta_-(2)\partial_1\Delta_-(1)\partial_2\Delta_-(2) = -\frac{1}{16}\left(\lambda^3 - \frac{\lambda}{2}\right) \tag{C.10}$$

$$\mathcal{I}_E = \int d\tau_1 d\tau_2 \, \Delta_+(1)\Delta_-(1+2)\Delta_+(2)\partial_1\Delta_-(1)\partial_2\Delta_-(2) = -\frac{1}{24}\left(\lambda^3 - \frac{\lambda}{2}\right) \tag{C.11}$$

$$\mathcal{I}_F = \int d\tau_1 d\tau_2 \, \Delta_-(1)\Delta_+(1+2)\Delta_-(2)\partial_1\Delta_-(1)\partial_2\Delta_-(2) = -\frac{1}{24}\lambda^3 \tag{C.12}$$

$$\mathcal{I}_G = \int d\tau_1 d\tau_2 \, \Delta_+(1)\Delta_+(1+2)\Delta_-(2)\partial_1\Delta_-(1)\partial_2\Delta_+(2) = \frac{1}{16}\left(\lambda^3 - \frac{\lambda}{6}\right) \tag{C.13}$$

$$\mathcal{I}_H = \int d\tau_1 d\tau_2 \, \Delta_-(1)\Delta_+(1+2)\Delta_-(2)\partial_1\Delta_-(1)\partial_2\Delta_+(2) = -\frac{1}{48}\left(\lambda^3 - \frac{\lambda}{2}\right) \tag{C.14}$$

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$$\mathcal{I}_I = \int d\tau_1 d\tau_2 d\tau_3\, \Delta_+(1+2)\Delta_+(1+2+3)\Delta_+(2+3)\partial_1\Delta_-(1)\partial_2\Delta_-(2)\partial_3\Delta_-(3)$$
$$= -\frac{i}{8}\left(\lambda^3 - \frac{\lambda}{2}\right) \tag{C.15}$$

$$\mathcal{I}_J = \int d\tau_1 d\tau_2 d\tau_3\, \Delta_+(1+2)\Delta_+(1+2+3)\Delta_-(2+3)\partial_1\Delta_-(1)\partial_2\Delta_-(2)\partial_3\Delta_-(3)$$
$$= -\frac{i}{16}\left(\lambda^3 - \frac{\lambda}{2}\right) \tag{C.16}$$

$$\mathcal{I}_K = \int d\tau_1 d\tau_2 d\tau_3\, \Delta_+(1+2)\Delta_-(1+2+3)\Delta_+(2+3)\partial_1\Delta_-(1)\partial_2\Delta_-(2)\partial_3\Delta_-(3)$$
$$= -\frac{i}{24}\left(\lambda^3 - \lambda\right) \tag{C.17}$$

$$\mathcal{I}_L = \int d\tau_1 d\tau_2 d\tau_3\, \Delta_-(1+2)\Delta_+(1+2+3)\Delta_-(2+3)\partial_1\Delta_-(1)\partial_2\Delta_-(2)\partial_3\Delta_-(3)$$
$$= -\frac{i}{24}\left(\lambda^3 - \frac{\lambda}{2}\right) \tag{C.18}$$

$$\mathcal{I}_M = \int d\tau_1 d\tau_2 d\tau_3\, \Delta_+(1+2)\Delta_+(1+2+3)\Delta_-(2+3)\partial_1\Delta_-(1)\partial_2\Delta_-(2)\partial_3\Delta_+(3)$$
$$= \frac{i}{48}\left(\lambda^3 + \frac{\lambda}{2}\right) \tag{C.19}$$

$$\mathcal{I}_N = \int d\tau_1 d\tau_2 d\tau_3\, \Delta_+(1+2)\Delta_-(1+2+3)\Delta_+(2+3)\partial_1\Delta_-(1)\partial_2\Delta_-(2)\partial_3\Delta_+(3)$$
$$= \frac{i}{48}\left(\lambda^3 - \frac{\lambda}{2}\right) \tag{C.20}$$

$$\mathcal{I}_O = \int d\tau_1 d\tau_2 d\tau_3\, \Delta_+(1+2)\Delta_-(1+2+3)\Delta_-(2+3)\partial_1\Delta_-(1)\partial_2\Delta_-(2)\partial_3\Delta_+(3)$$
$$= \frac{i}{24}\left(\lambda^3 - \frac{\lambda}{2}\right) \tag{C.21}$$

$$\mathcal{I}_P = \int d\tau_1 d\tau_2 d\tau_3\, \Delta_-(1+2)\Delta_+(1+2+3)\Delta_-(2+3)\partial_1\Delta_-(1)\partial_2\Delta_-(2)\partial_3\Delta_+(3)$$
$$= 0 \tag{C.22}$$

$$\mathcal{I}_Q = \int d\tau_1 d\tau_2 d\tau_3\, \Delta_+(1+2)\Delta_+(1+2+3)\Delta_-(2+3)\partial_1\Delta_-(1)\partial_2\Delta_+(2)\partial_3\Delta_-(3)$$
$$= 0 \tag{C.23}$$

$$\mathcal{I}_R = \int d\tau_1 d\tau_2 d\tau_3\, \Delta_+(1+2)\Delta_-(1+2+3)\Delta_+(2+3)\partial_1\Delta_-(1)\partial_2\Delta_+(2)\partial_3\Delta_-(3)$$
$$= 0 \tag{C.24}$$

$$\mathcal{I}_S = \int d\tau_1 d\tau_2 d\tau_3\, \Delta_-(1+2)\Delta_+(1+2+3)\Delta_-(2+3)\partial_1\Delta_-(1)\partial_2\Delta_+(2)\partial_3\Delta_-(3)$$
$$= \frac{i}{48}\lambda\,. \tag{C.25}$$

The diagrams corresponding to the integrals are depicted in figure (9).

All of these integrals were performed in the same way as the two loop diagram of appendix (B). They all essentially involve integrals over products of sign functions. Take
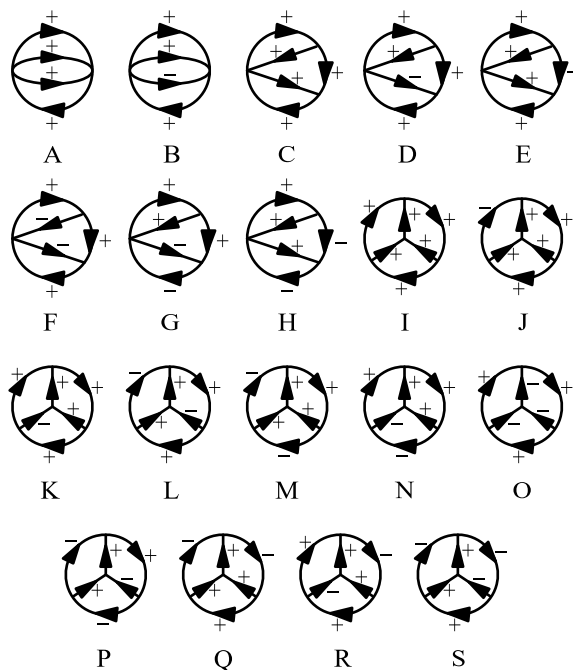
Figure 9: The basic non-trivial diagrams at three loops.

for instance $\mathcal{I}_M$. Using the definition of the τ-space propagators (3.7), we find more explicitly that

$$
\begin{aligned}
\mathcal{I}_M = \frac{i}{2^6 \pi^3} \int_1^\Lambda \frac{dp}{p} \int_1^\Lambda \frac{dq}{q} \int_1^\Lambda \frac{dk}{k} \big[&- \epsilon(p+q-\Lambda)\epsilon(p+q-k-\Lambda)\epsilon(q-k+1) \\
&+ \epsilon(p+q-\Lambda)\epsilon(p+q-k-1)\epsilon(q-k+1) \\
&+ \epsilon(p+q-\Lambda)\epsilon(p+q-k-\Lambda) - \epsilon(p+q-\Lambda)\epsilon(p+q-k-1) \\
&+ \epsilon(p+q-k-\Lambda)\epsilon(q-k+1) - \epsilon(p+q-k-1)\epsilon(q-k+1) \\
&- \epsilon(p+q-k-\Lambda) + \epsilon(p+q-k-1) \big].
\end{aligned}
\tag{C.26}
$$

Clearly, this kind of calculation is best done using a computer. The 19 integrals we have listed above can be seen as the building blocks out of which every other non-trivial three loop integral can be obtained quite easily. To illustrate this point, let us look at the integral corresponding to diagram A of figure (10).
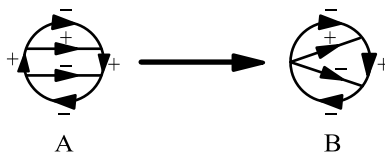

Figure 10: The integral of diagram (A) can be related to the one of diagram (B).

The reader may have noticed that this type of diagram did not appear in the list of

'building blocks' of figure (9). This is because an integral corresponding to this type of diagram can always be related to one corresponding to a diagram of the type of diagram B of figure (10). This is neatly illustrated with the case at hand. The integral corresponding to diagram A of figure (10) equals

$$
\begin{aligned}
\mathcal{I} = \int d\tau_1 d\tau_2 d\tau_3 \ \Delta_-(1)\Delta_-(2)\Delta_+(1+2+3)\partial_1\Delta_+(1)\partial_2\Delta_+(2)\partial_3\Delta_-(3) \\
+ \int d\tau_1 d\tau_2 d\tau_3 \ \Delta_+(1)\Delta_-(2)\Delta_+(1+2+3)\partial_1\Delta_-(1)\partial_2\Delta_+(2)\partial_3\Delta_-(3) \, .
\end{aligned}
\tag{C.27}
$$

Using the fact that

$$
\int d\tau_3 \ \Delta_\pm(1+2+3)\partial_3\Delta_\mp(3) = \pm i \Delta_\pm(1+2) \, ,
\tag{C.28}
$$

we can perform the integral over τ_3 and arrive at

$$
\begin{aligned}
\mathcal{I} = i \int d\tau_1 d\tau_2 \ \Delta_-(1)\Delta_-(2)\Delta_+(1+2)\partial_1\Delta_+(1)\partial_2\Delta_+(2) \\
+ i \int d\tau_1 d\tau_2 \ \Delta_+(1)\Delta_-(2)\Delta_+(1+2)\partial_1\Delta_-(1)\partial_2\Delta_+(2) \, ,
\end{aligned}
\tag{C.29}
$$

which indeed corresponds to diagram B of figure (10). This diagram does not correspond to any of the building blocks of figure (9) either, but comparison with the ones that do, shows that we can write

$$
\mathcal{I} = i \left(\mathcal{I}_E + \mathcal{I}_G \right) \, .
\tag{C.30}
$$

This illustrates how other integrals one needs to perform can be related to the building blocks.

The advantage of this method is that it can, in principle, easily be generalized to higher loops. First of all, superspace techniques can just as easily be applied to higher loop diagrams. The τ-space integral one ends up with will always be a generalization of eqs. (B.3) and (C.26). The general procedure explained in this and the previous appendix to handle these kinds of integrals can still be applied at higher loops. These integrals will always be expressible in terms of logarithms and polylogarithms. At l loops, a general term will be of the form[7]

$$
\text{Li}_n \left(r(\Lambda) \right) \ \ln^{l-n} \left(s(\Lambda) \right), \qquad n \le l,
\tag{C.31}
$$

where r and s are rational functions of Λ. The n-th order polylogarithm Li_n is defined as

$$
\text{Li}_n(x) = \int_0^x dz \ \frac{\text{Li}_{n-1}(z)}{z} = \sum_{k=1}^\infty \frac{x^k}{k^n} \, , \quad |x| \le 1 \, ,
\tag{C.32}
$$

---

[7]Eq. (C.31) only serves as a rough indication of the general form. More concretely, not all factors of ln need to have the same argument and there might be more polylogarithms involved in the same term. The important point is that all powers of ln and all orders of the polylogarithms involved add up to l.

and Li_2 is defined in eq. (B.8). The last equality in eq. (C.32) is only valid for $|x| \leq 1$, as indicated, but the integral representation can be used as a definition of Li_n on the whole (cut) complex plane. Since we are only interested in the behavior of eq. (C.31) for $\Lambda \gg 1$, we can use identities like [39][8]

$$\text{Li}_n(-x) + (-1)^n \, \text{Li}_n\left(-\frac{1}{x}\right) = -\frac{1}{n!} \ln^n(x) + 2 \sum_{k=1}^{[n/2]} \frac{\text{Li}_{2k}(-1)}{(n-2k)!} \, \ln^{n-2k}(x), \quad \text{(C.33)}$$

where $[n/2]$ is the greatest integer contained in $n/2$ and

$$\text{Li}_n(-1) = \left(2^{1-n} - 1\right) \zeta(n), \tag{C.34}$$

to write the asymptotic behavior of (C.31) purely in terms of $\lambda \sim \ln(\Lambda)$. In the end we still arrive at a polynomial in λ, as desired.

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[8]For $n = 2$ and 3, this expression reduces to eqs. (C.3) and (C.5), respectively.

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